2023 CALCULUS

Questions ○ Memos ○ Diagnostic Report

THE ANSWER SERIES

Your Key to Exam Success

QUESTION 7 66%

7.1 Determine $f'(x)$ from first principles if $f(x) = -4x^2$ (5)

83%

Memo

7.1

$$f(x) = -4x^2$$

$$\therefore f(x+h) = -4(x+h)^2$$

$$= -4(x^2 + 2xh + h^2)$$

$$= -4x^2 - 8xh - 4h^2$$

$$\therefore f(x+h) - f(x) = -8xh - 4h^2$$

$$\therefore \frac{f(x+h) - f(x)}{h} = -8x - 4h$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

def of a derivative

$$= \lim_{h \to 0} (-8x - 4h)$$

$$= \mathbf{-8x}$$

OR: $f'(x) = \lim_{h \to 0} \frac{-4(x+h)^2 - (-4x^2)}{h}$

$$= \lim_{h \to 0} \frac{-4x^2 - 8xh - 4h^2 + 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{-8xh - 4h^2}{h}$$

$$= \lim_{h \to 0} (-8x - 4h)$$

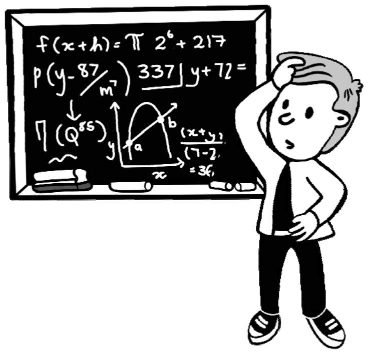
$$= -8x$$

Common Errors and Misconceptions

(a) In **Q7.1** some candidates made the following **notational errors**:

$\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or left out the limit of h altogether.

They lost a mark for these errors. Other candidates made errors in substitution.



7.2 Determine:

79%

7.2.1 $f'(x)$ if $f(x) = 2x^3 - 3x$ (2)

7.2.2 $D_x\left(7.\sqrt[3]{x^2} + 2x^{-5}\right)$ (3)

Memo

7.2.1 $f(x) = 2x^3 - 3x$

$\therefore$ $\mathbf{f'(x) = 6x^2 - 3}$ ✓

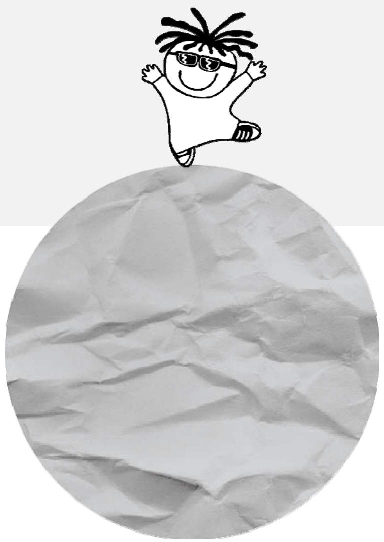
7.2.2 $D_x\left(7.x^{\frac{2}{3}} + 2x^{-5}\right)$

$= 7.\frac{2}{3}x^{-\frac{1}{3}} + 2.-5x^{-6}$

$= \mathbf{\frac{14}{3}.x^{-\frac{1}{3}} - 10x^{-6}}$ ✓

$= \mathbf{\frac{14}{3x^{\frac{1}{3}}} - \frac{10}{x^6}}$ ✓

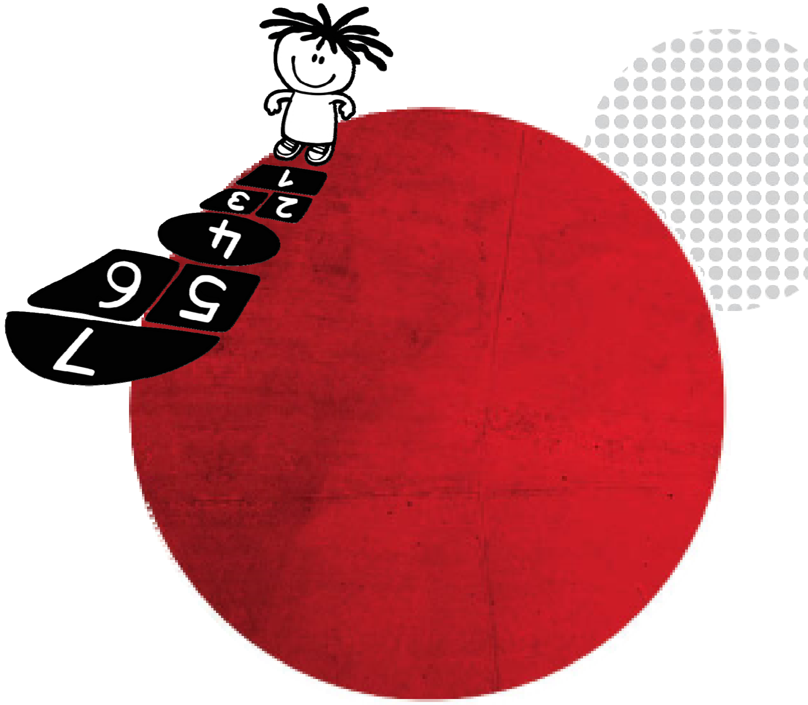
$= \mathbf{\frac{14}{3\sqrt[3]{x}} - \frac{10}{x^6}}$ ✓



Common Errors and Misconceptions

(b) In **Q7.2.2** candidates incorrectly changed

$7.\sqrt[3]{x^2}$ to $7x^{\frac{3}{2}}$ instead of $7x^{\frac{2}{3}}$.



7.3 For which values of x will the tangent to $f(x) = -2x^3 + 8x$ have a positive gradient? (3)

16%

Memo

7.3

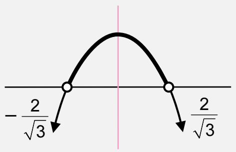
> The tangent to a graph:
> The gradient of the tangent to a graph f is the derivative, $f'(x)$

$f(x) = -2x^3 + 8x$

$\therefore$ The gradient of the tangent $= f'(x) = -6x^2 + 8$

$f'(x) > 0 \Rightarrow -6x^2 + 8 > 0$

The graph of $y = f'(x)$:



The roots: $-6x^2 + 8 = 0$

$\therefore 6x^2 - 8 = 0$

$\therefore 6x^2 = 8$

$\therefore x^2 = \frac{4}{3}$

$\therefore x = \pm\frac{2}{\sqrt{3}}$

$\therefore -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$ ◄

[or $-1{,}15 < x < 1{,}15$ ◄]

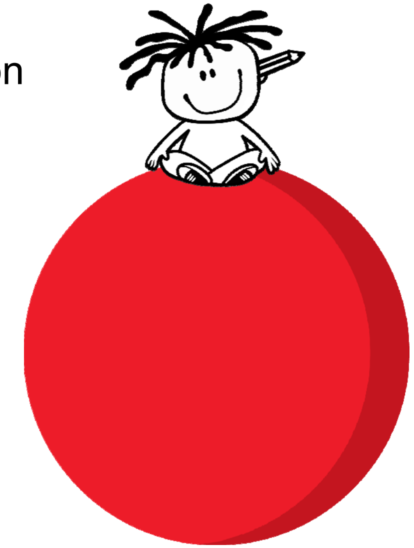
Common Errors and Misconceptions

(a) When answering **Q7.3**, many candidates calculated the first derivatives correctly. However, they did not understand how to solve the **inequality** $x^2 > \frac{4}{3}$, partly because the square root of $\frac{4}{3}$ is not a rational number and partly because of a lack of understanding of how to work with inequalities.

QUESTION 7: Suggestions for Improvement

(a) Emphasis should be placed on the use of the correct **notation** when determining the derivative, either when using first principles or the rules.

(b) Teachers should revise the rules of ***exponents*** and ***surds*** when changing an expression into differentiable format.



(c) The concepts of ***gradient of tangent*** and ***point of inflection*** must be understood. Teachers should explain the difference between the two concepts.

(d) **Integration** and re-emphasis of **algebraic concepts**, viz fractions, factorising, inequalities and exponential rules, should be emphasised when working with *calculus*.

QUESTION 8 47%

Given: $f(x) = -x^3 + 6x^2 - 9x + 4 = (x - 1)^2(-x + 4)$

8.1 Determine the coordinates of the turning points of f. (4)

63%

Memo

8.1 $f(x) = -x^3 + 6x^2 - 9x + 4 = (x - 1)^2(-x + 4)$

At the turning points, $\mathbf{f'(x) = 0}$

$\therefore f'(x) = -3x^2 + 12x - 9 = 0$

$\div(-3)$: $\therefore x^2 - 4x + 3 = 0$

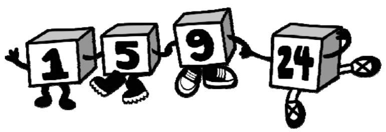
$\therefore (x - 1)(x - 3) = 0$

$\therefore x = 1$ or 3

$f(1) = (1 - 1)^2(-1 + 4)$
$= 0$

& $f(3) = (3 - 1)^2(-3 + 4)$
$= 4$

$\therefore$ **Turning points are (1; 0) and (3; 4)** ◂



Common Errors and Misconceptions

(a) Some candidates did not realise that the function in **Q8.1** was already factorised for them. They expanded the factors and equated to the left side. Instead of getting zero, they created another function. These candidates also solved for the ***x*-intercepts** rather than the ***x*-coordinates of the turning points** first.

(b) In **Q8.1** candidates did not equate the derivative to 0 explicitly. Some candidates only worked out the x values without calculating the corresponding y values. This led to an incorrect graph in **Q8.2**. Other candidates attempted to use the quadratic equation principle of $x = -\frac{b}{2a}$ to calculate the x-coordinate of the turning point which does not lead to the correct answer in a cubic function.

8.2 **54%** Draw a sketch graph of f. Clearly label all the intercepts with the axes and any turning points. (4)

8.3 **18%** Use the graph to determine the value(s) of k for which $-x^3 + 6x^2 - 9x + 4 = k$ will have three real and unequal roots. (2)

Memo

8.2 **Y-intercept:** $f(0) = 4$

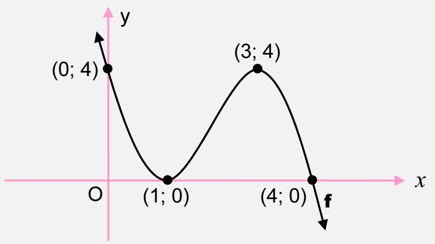
$(x = 0)$ $\therefore$ $(0; 4)$

X-intercepts: $(x - 1)^2(-x + 4) = 0$

$(y = 0)$ $\therefore$ $x = 1$ ('twice': this is also a t.pt)

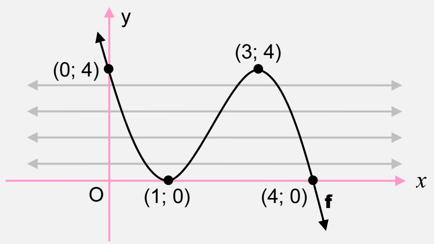
or $x = 4$

$\therefore$ $(1; 0)$ and $(4; 0)$



8.3 **0 < k < 4** ✓

y = k could be (the eqn of) any of these **grey** lines. They all cut f at 3 distinct points.



Common Errors and Misconceptions

(c) Many candidates did not label the graph in **Q8.2** even though it was explicitly asked of them to do so. Other candidates could not draw the cubic function and drew a version of a parabola. A common mistake was not indicating the y-intercept.

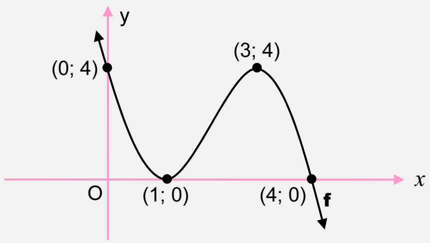
(d) In **Q8.3** most candidates did not use graphical interpretation to answer this question but rather used quadratic function theory and substituted in $b^2 - 4ac$ which was **incorrect**.

8.4 **42%** The line $g(x) = ax + b$ is the tangent to f at the point of inflection of f. Determine the equation of g. (6)

8.5 **42%** Calculate the value of θ, the acute angle formed between g and the x-axis in the first quadrant. (2)

[18]

Memo



8.4 At the point of inflection: $f''(x) = 0$

$\therefore f''(x) = -6x + 12 = 0$

$\therefore -6x = -12$

$\therefore x = 2$

$f(2) = 2$

$\therefore$ **The point of inflection is (2; 2)**

The gradient of the tangent to f,

$a = f'(2) = -3(2)^2 + 12(2) - 9$

$= -12 + 24 - 9$

$= 3$

Substitute $a = 3$ and $(2; 2)$ in $y = ax + b$:

$\therefore 2 = (3)(2) + b$

$\therefore b = -4$

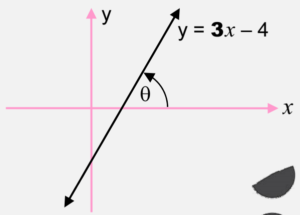
(or $y - y_1 = m(x - x_1)$

$\therefore y - 2 = 3(x - 2)$, etc.)

$\therefore$ The eqn of g: $\mathbf{y = 3x - 4}$ ◄

8.5 $\tan\theta = 3$ . . . *the gradient of g*

$\therefore \theta = \mathbf{71{,}57^\circ}$ ◄



Common Errors and Misconceptions

(e) In **Q8.4** many candidates substituted into $m = \frac{\Delta y}{\Delta x}$ to calculate a gradient rather than using calculus methods to solve for the **gradient**.

(f) Many candidates equated $m = \tan\theta$ where the gradient they had calculated was a negative value. The link that the gradient was positive was not recognised by the candidates.

QUESTION 8: Suggestions for Improvement

(a) Learners should be taught to determine the properties of a graph from Grade 10 to 12 in a progressive manner. The defining property of a turning point having a zero gradient is a way to describe a turning point. Teachers need to prove this **link** between the definition of a turning point and the Grade 11 concept of determining the axis of symmetry to calculate the x-coordinate of the turning point.

(b) When teaching *factorisation of third-degree polynomials*, teachers should include examples where there is only one real root.

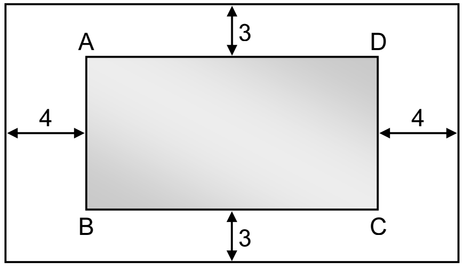
(c) Teachers should continue to teach graphical interpretation in cubic graphs as a follow on from the interpretation taught in Grade 10 and 11.

(d) The **concept** of the point of inflection needs to be taught explicitly.

(e) The application of Calculus lends itself to many applications. Teachers need to expose learners to a wide variety of questions, which include **integration of topics** including ***analytical geometry***, ***measurement*** **and** ***trigonometry***.

QUESTION 9 26%

The diagram below represents a printed poster. Rectangle ABCD is the part on which the text is printed. This shaded area ABCD is 432 cm^2 and AD = x cm. ABCD is 4 cm from the left and right edges of the page and 3 cm from the top and bottom of the page.



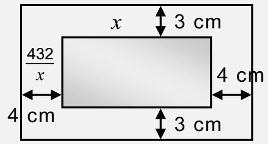
9.1 Show that the total area of the page is given by: **20%**

$$A(x) = \frac{3\,456}{x} + 6x + 480$$ (3)

9.2 Determine the value of x such that the total area of the page is a minimum. **31%** (3) **[6]**

Memo

9.1 The area of rectangle ABCD:



$x \times AB = 432\ cm^2$

$\therefore\ AB = \frac{432}{x}$ cm

$\therefore$ The total area $= (x + 8)\left(\frac{432}{x} + 6\right)$

$= 432 + 6x + \frac{3\,456}{x} + 48$

$= \boldsymbol{\frac{3\,456}{x} + 6x + 480}$ ✓

9.2 The total area, A $= 3\,456x^{-1} + 6x + 480$

For minimum value, $\frac{dA}{dx} = 0$

$\therefore\ -3\,456x^{-2} + 6 = 0$

$\therefore\ -\frac{3\,456}{x^2} = -6$

$\times(-x^2)$ $\therefore\ 3\,456 = 6x^2$

$(\div 6)$ $\therefore\ 576 = x^2$

$\therefore\ \boldsymbol{x = 24}$ **cm** ... $x > 0$ ✓

Common Errors and Misconceptions

(a) The vast majority of the candidates did not attempt this question because they were unable to formulate the equation required in **Q9.1**.

(b) Many candidates did not use the given equation in **Q9.1** to answer **Q9.2**.

(c) Some common errors included:

- Candidates simplified the equation required in **Q9.2** to $x^{-2} = \frac{1}{576}$ and then **incorrectly** concluded that $x = \frac{1}{24}$.
- Candidates calculated that $x^2 = 576$ and then concluded that $x = -24$ rather than $x = 24$.

QUESTION 9: Suggestions for Improvement

(a) Learners appear to be dependent on the formulae being given when solving *optimisation* problems.

It is advisable that learners interrogate **the *optimum* function** even when it is given in a question.

This should help their conceptual development.

(b) Teachers should ensure that there is enough time for learners to understand the application of Calculus fully.

(c) Reading for understanding should be ongoing if learners are to improve their responses to word problems.