Further Studies Mathematics

BOOK 2

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& Ingrid Zlobinsky-Roux

GRADE

10-12

ISC

EXTENDED
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Further Studies Mathematics

Book 2: Extended Level

Statistics, Finance & Mathematical Modelling, Matrices & Graph Theory

M. Buchanan, G. Esterhuyse, C. Fourie, N. Jakins & I. Zlobinsky-Roux

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CONTENTS of Book 2: Elective Modules

STATISTICS

IMPORTANT TO NOTE

Advanced Programme Mathematics is not an independent subject.

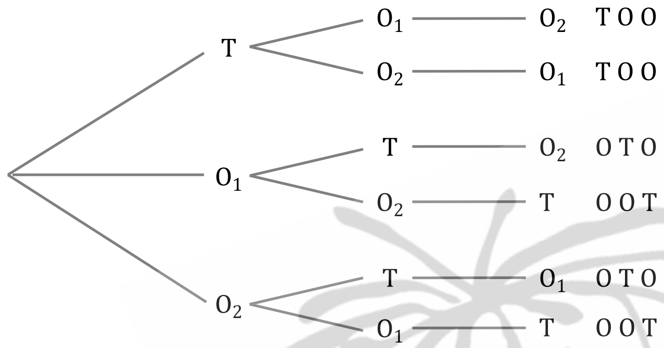
Knowledge and understanding of the core mathematics curriculum is a prerequisite as each module of the Advanced Mathematics Programme is introduced. In TAS AP study guides, we have not wanted to duplicate the development and mastering of core maths concepts where these are dealt with timeously in the core curriculum, as noted in the standard pace setters. Learners and teachers should therefore incorporate their core maths resources as part of their work for AP Maths.

FINANCE & MODELLING

MATRICES & GRAPH THEORY

Arranging n objects with m_1, m_2, etc. identical objects (using all objects)

We can again use a tree diagram to determine the number of arrangements of the letters from the word TOO



We expected 6 arrangements but some of them are identical owing to the two identical O's. There are now only 3 different arrangements.

$$\text{Number of arrangements} = \frac{3!(\text{number of letters})}{2!(\text{number of O's})} = 3$$

Worked Example 14

Calculate how many ways the letters from the word CURRICULUM can be arranged if:

(a) there are no restrictions.

(b) the three U's must be at the beginning of the word.

(c) the three U's must be together.

Solutions

(a) $\dfrac{10!(\text{number of letters})}{2!(\text{number of C's}) \times 3!(\text{number of U's}) \times 2!(\text{number of R's})} = \mathbf{151\ 200}$

(b) $\dfrac{7!(\text{number of letters excluding U's})}{2!(\text{number of C's}) \times 2!(\text{number of R's})} = \mathbf{1\ 260}$

(c) $\dfrac{8!(\text{number of letters with U's as 1 letter})}{2!(\text{number of C's}) \times 2!(\text{number of R's})} = \mathbf{10\ 080}$

Exercise STAT 1.5

(Solutions on Stat.98)

1. In how many ways can the letters of the following names be arranged?
   (a) LIESEL (b) NATASHA
   (c) JEANETTE (d) KRASSNOKUTSKI

2. There are 9 different books on a bookshelf, one of which is a dictionary and one an atlas. In how many ways can the books be arranged on the shelf if the dictionary and the atlas are to be next to each other?

3. 10 coloured beads, 4 red, 3 blue, 1 orange, 1 white and 1 green are to be threaded onto a wire. Find the number of ways in which this can be done.

4. How many different arrangements of letters are there in the word COMBINATION if the "word" must start and end with an "O"?

5. In how many ways can 8 people be seated in a row if:
   (a) there are no restrictions?
   (b) they consist of 3 couples and 2 single people and the couples want to sit with their partners?

6. In how many ways can 7 guests on a television talk show be seated in a row of 7 seats?

7. Liesel is tidying her room. She has 10 different English books and 3 different Afrikaans books. In how many ways can she arrange the books on her bookshelf if she wants to keep the English books and the Afrikaans books together?

8. 4 houses compete in the inter-house Mathematics Olympiad. In how many ways could they be placed in 1^{st}, 2^{nd}, 3^{rd} and 4^{th} places?

9. Four couples see the show "Phantom of the Opera" together. In how many different ways can they be seated if:
   (a) the couples sit together?
   (b) all the ladies sit together and all the men sit together?

10. In how many ways can 6 different pot plants be arranged on a windowsill?

(c) $P(2{,}1 < X < 3{,}4)$ = area under the curve from $x = 2{,}1$ to $x = 3{,}4$
$= 0{,}25(3{,}4 - 2{,}1)$
$= \mathbf{0{,}325}$

(d) The median is defined as the value m which divides the area into two parts each with area $= \frac{1}{2}$.

$$\frac{1}{4} \times (m - 1) = \frac{1}{2}$$
$$\therefore\ m - 1 = 2$$
$$\therefore\ \boldsymbol{m} = \mathbf{3}$$

Worked Example 31

Given the following p.d.f: $f(x) = \begin{cases} \frac{1}{20} & \text{for } 0 < x < 20 \\ 0 & \text{otherwise} \end{cases}$

Determine:

(a) $P(X < 5)$

(b) $P(X > 12)$

(c) $P(2 < X < 13)$

(d) the median of this distribution.

Solutions

(a) $P(X < 5) = 5 \times \frac{1}{20} = \mathbf{\frac{1}{4}}$

(b) $P(X > 12) = \frac{20-12}{1} \times \frac{1}{20} = \mathbf{\frac{2}{5}}$

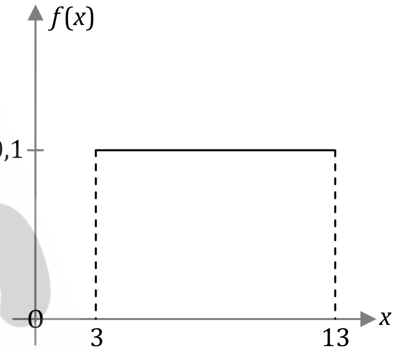
(c) $P(2 < X < 13) = (13 - 2) \times \frac{1}{20} = \mathbf{\frac{11}{20}}$

(d) $\frac{1}{20} \times (m - 0) = \frac{1}{2}$
$\therefore\ \boldsymbol{m} = \mathbf{10}$

Exercise STAT 2.7

(Solutions on Stat.110)

1. The following is a graphical representation of a rectangular p.d.f.



   (a) Write down the equation of the p.d.f.

   (b) Determine $P(5{,}2 < X < 7{,}1)$

   (c) Determine $P(X > 8{,}7)$

   (d) Find the median of this distribution

2. The p.d.f. of a random variable is: $f(x) = \begin{cases} \frac{1}{k} & \text{for } 2 < x < 10 \\ 0 & \text{otherwise} \end{cases}$

   Determine:

   (a) the value of k.

   (b) $P(X < 3{,}5)$

   (c) $P(X > 7{,}5)$

   (d) $P(5{,}5 < X < 6{,}5)$

   (e) the median of this distribution

3. The p.d.f. of a random variable is: $f(x) = \begin{cases} \frac{1}{7} & \text{for } k < x < 10 \\ 0 & \text{otherwise} \end{cases}$

   Determine:

   (a) the value of k.

   (b) $P(X > 9{,}3)$

   (c) $P(X < 5{,}2)$

   (d) $P(6{,}4 < X < 7{,}1)$

   (e) the median of this distribution

In general, when standardising values, the continuity corrections are as follows:

$P(X = 20) \rightarrow P(19{,}5 < X < 20{,}5) = P\left(\frac{19{,}5 - \mu}{\sigma} < Z < \frac{20{,}5 - \mu}{\sigma}\right)$

$P(X < 20) \rightarrow P\left(Z < \frac{19{,}5 - \mu}{\sigma}\right)$

$P(X \leq 20) \rightarrow P\left(Z \leq \frac{20{,}5 - \mu}{\sigma}\right)$

$P(X > 20) \rightarrow P\left(Z > \frac{20{,}5 - \mu}{\sigma}\right)$

$P(X \geq 20) \rightarrow P\left(Z \geq \frac{19{,}5 - \mu}{\sigma}\right)$

The ≤ and ≥ are only of significance with the continuity corrections. As far as the probabilities are concerned, it does not matter whether we say $P(Z \geq a)$ or $P(Z > a)$.

Procedure for the Normal approximation to the Binomial:

❶ Check to see if the normal approximation can be used ($np \geq 5$ and $nq \geq 5$).

❷ Find the mean and the standard deviation $\left(\mu = np;\ \sigma = \sqrt{npq}\right)$.

❸ Write the problem in probability notation, using X.

❹ Rewrite the problem using the continuity correction factor.

❺ Find the corresponding z-scores where $z = \frac{\bar{x} - \mu}{\sigma}$.

❻ Find the solution.

Worked Example 12

Research has shown the probability of a person belonging to blood group O is 0,45.

(a) Use the binomial mass function to determine the probability that out of a group of 100 people, exactly 30 will belong to blood group O.

(b) Use the normal approximation to determine the probability that, out of a group of 100 people, exactly 30 will belong to blood group O.

(c) Determine the probability that out of a group of 100 people, at least 30 will belong to blood group O.

(d) Determine the probability that out of a group of 100 people, fewer than 30 will belong to blood group O.

Solutions

(a) $P(X = x) = \binom{100}{x} 0{,}45^x\, 0{,}55^{100 - x}$

$\therefore\ P(X = 30) = \binom{100}{30} 0{,}45^{30}\, 0{,}55^{70}$

$\approx$ **0,0007757**

(b) ❶ $p = 0{,}45$; $q = 1 - p = 0{,}55$; $n = 100$

Can a normal approximation be used?

$np = 100 \times 0{,}45 = 45 > 5$

$nq = 100 \times 0{,}55 = 55 > 5$

Yes it can be used.

❷ $u = np = 45$

$\sigma = \sqrt{npq} = \sqrt{100 \times 0{,}45 \times 0{,}55} \approx 4{,}9749$

❸ $P(X = 30)$

❹ $P(29{,}5 < X < 30{,}5)$

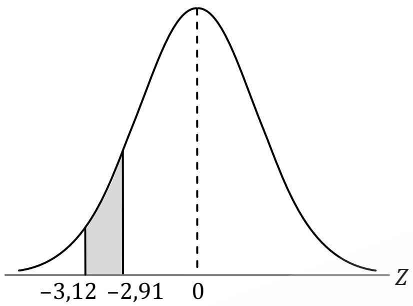
❺ $P\left(\dfrac{29{,}5-45}{4{,}9749} < Z < \dfrac{30{,}5-45}{4{,}9749}\right)$

$\therefore\ P(-3{,}12 < Z < -2{,}91)$

❻ $H(3{,}12) - H(2{,}91)$

$= 0{,}4991 - 0{,}49819$

$= \mathbf{0{,}00091}$



(c) We have already established that a normal approximation can be used and we have determined μ and σ.

$P(X \geq 30) \ \rightarrow\ P(X > 29{,}5)$

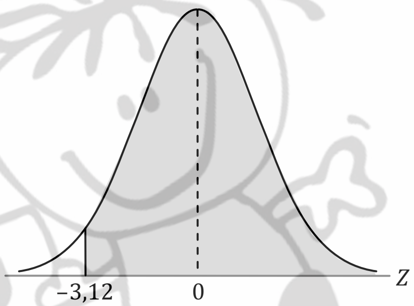
$= P\left(Z > \dfrac{29{,}5-45}{4{,}9749}\right)$

$= P(Z > -3{,}12)$

$= 0{,}5 + H(3{,}12)$

$= 0{,}5 + 0{,}4991$

$= \mathbf{0{,}9991}$



(d) We have already established that a normal approximation can be used and we have determine μ and σ.

$P(X < 30) \ \rightarrow\ P(X < 29{,}5)$

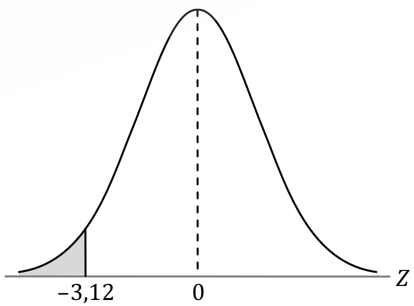
$= P\left(Z < \dfrac{29{,}5-45}{4{,}9749}\right)$

$= P(Z < -3{,}12)$

$= 0{,}5 - H(3{,}12)$

$= 0{,}5 - 0{,}4991$

$= \mathbf{0{,}0009}$



Exercise STAT 3.4

(Solutions on Stat.122)

1. It is known that 20% of all people are left-handed. In a statistics class of 50 students, determine:
   (a) the probability that at most 2 are left-handed by using the binomial distribution.
   (b) the probability that at most 2 are left-handed by using a normal approximation to the binomial distribution.
   (c) the probability that at least 10 are left-handed.
   (d) the probability that more than 7 are left-handed.

2. Nabeel notes that in his suburb, 72% of households belong to a certain security company. Find the probability that, in a random sample of 50 households in the suburb:
   (a) at least 30 belong to that security company.
   (b) more than 25 belong to that security company.
   (c) at most 35 belong to that security company.

3. A cell phone manufacturing company is testing their cell phones. A random sample of 400 cell phones is selected. Determine the probability that more than 5 cell phones are defective if it is known that 1,5% of all manufactured cell phones are defective.

4. The death rate from pneumonic plague is about 11%, thanks to the availability of antibiotics that can treat it. Determine the probability that out of 325 patients who contract the disease, at most 25 will die from the disease.

5. At a particular coffee shop, the probability that a customer will buy a cappuccino is 66%. Determine the probability that out of a random sample of 150 customers, at least 87 will buy a cappuccino.

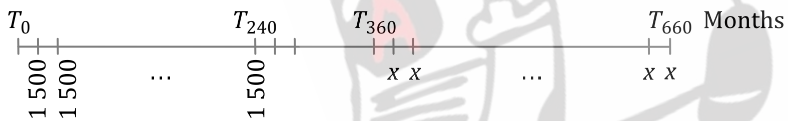
6. In the first year statistics class, 60% of students are men and 40% are women. In a random sample of 50 students, what is the probability that more than half are women?

A Retirement Plan is often referred to as a **Deferred Annuity**. It consists of 2 phases: the Investment phase (when a person makes regular deposits into the retirement fund) and the Income phase (when regular withdrawals are made, typically after retirement). Most of these plans entrust the growth of the investment to experts who invest the fund money in order to generate the best returns. However, in this curriculum, we will stick to investment methods with fixed growth specified.

Worked Example 11

Mr Maseko set up an investment strategy whereby he deposited R1 500 (at the end of each month), into a bank account earning interest at a rate of 7,2% p.a., compounded monthly. After 20 years, Mr Maseko stopped making any further payments, but left the accumulated amount in the account. At his retirement 10 years later, he started to live off his investment by making withdrawals (at the end of each month) for 25 years. Calculate the value of his monthly withdrawals.

Solution



We use the Future Value formula for the (20×12) payments into the account. This amount then earns interest for (10×12) months, at which point it becomes the Present Value for Mr Maseko's (25×12) withdrawals.

$$1 + \frac{0{,}072}{12} = \frac{503}{500} \quad \rightarrow \text{MEM A}$$

$$\text{At } T_{360}: \quad \frac{1\,500\left[A^{240} - 1\right]}{A - 1}.A^{120} = \frac{x\left[1 - A^{-300}\right]}{A - 1}$$

$$\therefore\ 1\,641\,333{,}649 = x(138{,}9682\ldots)$$

$$\therefore\ \boldsymbol{x \approx \textbf{R11 810,85 per month}}$$

Exercise FIN 2.4

(Solutions on Fin.57)

1. James took up a promotional offer in which he was able to buy electronic equipment for R40 000, and only start making repayments at the end of 6 months with the loan amortised at the of 2 years. The loan agreement with the bank involved interest charged at 14% p.a., compounded monthly. Calculate the value of James' monthly payments.

2. Jenni started a nursery school by taking a "start-up business" loan of R100 000 from a bank. She only needed to start the repayments at the end of 2 years, but was charged interest of 8% p.a., compounded monthly on the outstanding balance. Jenni was contracted to clear the debt in equal monthly instalments (at the end of each month) by the end of 5 years from when she obtained the loan. Calculate the value of her monthly payments.

3. Andy secures a loan of Rx that needs to be paid off over a period of five years from the granting of the loan. However, he will only start paying off the loan four months after it is granted (at the end of each month) in equal instalments of R680. Interest on the loan is charged at 6,8% per annum, compounded monthly. Calculate the value of the loan.

4. Mandla started a savings plan by depositing R600 (at the end of each month) into a bank account earning interest at a rate of 7,6% p.a., compounded monthly. After 15 years of making these deposits, Mandla had extra expenses, so stopped making any further payments, but left the accumulated amount in the account. At his retirement 25 years after he stopped making payments into the fund, he started to live off his investment by making withdrawals (at the end of each month) for 20 years. Calculate the value of his monthly withdrawals.

Change in payments

A person's financial situation may change, allowing them to pay a different amount into an investment.

Worked Example 8

Chris took a loan of R70 000 with interest charged at 15% p.a., compounded monthly. For 5 years he paid R1 600 at the end of each month, but then he received an increase in salary enabling him to repay R2 100 per month. Determine how long it took Chris to amortise the loan.

Solution

$1 + \frac{0{,}15}{12} = \frac{81}{80}$ → MEM A

We consider 2 phases of the repayments:

Phase 1: with payments from T_1 to T_{60} of R1 600

Phase 2: with payments from T_{61} to T_n of R2 100. This annuity amount needs to be brought back to a Present Value.

At T_0 for Phase 1:

$$P = \frac{1\,600\left[1 - A^{-60}\right]}{A - 1}$$
$$= 67\,255{,}346\,87 \quad \rightarrow \text{MEM B}$$

For Phase 2:

$$P = \frac{2\,100\left[1 - A^{-n}\right]}{A - 1} \times A^{-60}$$
$$= \left[1 - A^{-n}\right](79\,727{,}35\,723)$$

MEM C

$$\therefore\ 70\,000 = B + \left[1 - A^{-n}\right] \times C$$
$$\therefore \left[1 - A^{-n}\right] \times C = 2\,744{,}653\ldots$$
$$\therefore \left[1 - A^{-n}\right] = 0{,}0\,344\ldots$$
$$\therefore\ A^{-n} = 0{,}96\,557\ldots$$
$$\therefore\ -n = \log_A[ANS] = -2{,}82\ldots$$

SOLVE: Appendix 1

∴ After 60 months (with R1 600 payments) he paid 2 payments of R2 100 plus a third final payment. **63 months = 5 years 3 months.**

Exercise FIN 3.3

(Solutions on Fin.63)

1. Aarav plans a holiday to Durban and so opens a bank account earning interest at a rate of 7% p.a., compounded monthly. He commits to depositing R500 into this account at the end of each month. After 6 months, the interest rate changes to 7,25% p.a., compounded monthly. Calculate how much money will be in this account at the end of 2 years.

2. Anthea took a loan from a bank, charging interest at a rate of 10% p.a., compounded monthly. She decided that she could spare R10 000 at the end of each month from her salary, to pay off the loan. The bank agreed to give her 20 years to do this.
   (a) Calculate how large a loan Anthea took.
   (b) At the end of 9 months, the interest rate increased to 10,5% p.a., compounded monthly. Calculate the increase in Anthea's monthly payments.
   (c) Another 15 months later (i.e. 2 years from when the loan was granted) the interest rate jumped up by 1% and Anthea was not able to pay the increased monthly payments. Determine what extension in repayment time she needed to negotiate with the bank.
   (d) Consider the feasibility of Anthea been given an extension if the interest rate had increased by 2%.

3. Emily borrows R50 000 with interest charged at 11,5% p.a., compounded monthly. She plans to pay R1 000 at the end of each month for the first year, R1 200 per month for the second year, then R1 500 per month for the remaining time it will take to amortise the loan. Calculate how many payments Emily will make.

Exercise MOD 4.1

(Solutions on Mod.53)

1. Translate each of the following into symbolic language.
   (a) The thirty-fourth term has a value of 87.
   (b) The second term is obtained when 8 is added to the first term.
   (c) The thirteenth term is obtained when 3 is subtracted from double the twelfth term.
   (d) The n^{th} term is obtained by multiplying the preceding term by 3, after which 1 is added.

2. For each of the following situations:
   - generate the first 5 terms
   - give the recursive formula.

   (a) Start at 75 and subtract 5.
   (b) Start at 11, multiply by two and add 3.
   (c) Start at 144, halve the number and then add 12.
   (d) Start at –1, double the number and then subtract 1.

3. Generate the first five terms from the following recursive formulae:
   (a) $T_n = T_{n-1} + 8$ with $T_1 = -6$
   (b) $T_n = -3.T_{n-1} - 2$ with $T_1 = 3$
   (c) $T_n = \frac{1}{2}.T_{n-1} + 6$ with $T_1 = -8$
   (d) $T_n = 2.T_{n-1} - 4$ with $T_1 = 3$
   (e) $T_n = -\frac{1}{4}.T_{n-1} - 2$ with $T_1 = 24$
   (f) $T_n = T_{n-1} + n + 3$ with $T_1 = 0$

4. For each of the following sequences:
   - Write down an explicit formula,
   - Write down a recursive formula (by inspection), and
   - Determine the value of the 17^{th} term.

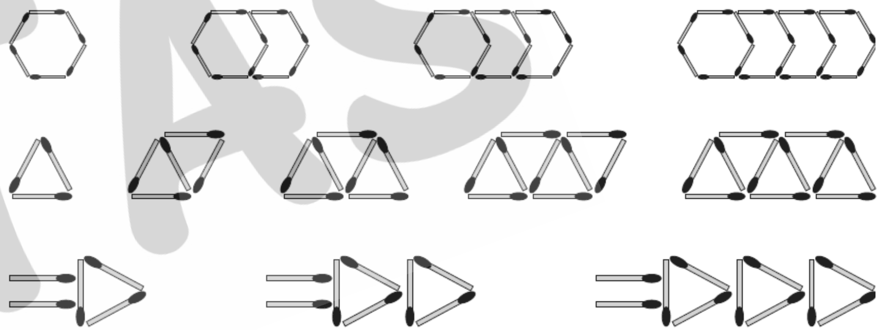
   (a) 2; 5; 8; 11; …
   (b) 4; 16; 64; 256; …
   (c) 1; –3; 9; –27; …
   (d) –4; –1,5; 1; 3,5; …
   (e) 1; 6; 11; 16; 21; …
   (f) 0,04; 0,2; 1; 5; …

5. For each of the following sequences:
   - Determine the recursive formula in the form $T_n = k.T_{n-1} + c$, and
   - Calculate the value of the twelfth term.

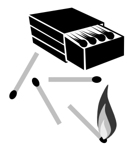
   (a) 2; 4; 10; 28; …
   (b) 3; 5; 9; 17; …
   (c) 48; 20; 6; –1; …
   (d) 6; –12; 60; –228; …
   (e) 225; 81; 33; 17; …
   (f) $\frac{1}{2}$; 6; –16; 72; …

6. In each of the figures drawn below, the next pattern is created by adding more matchsticks. Using the figures:

   (a)
   (b)
   (c)

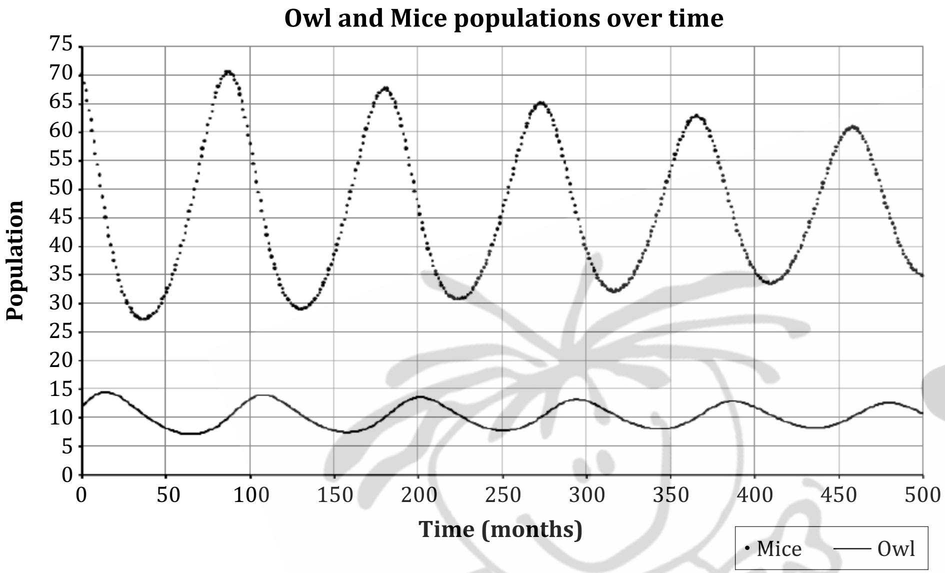


   (i) Find the number of matchsticks in the 5^{th} and 6^{th} patterns.
   (ii) Write down a recursive formula representing the number of matchsticks in any pattern.
   (iii) Write down an explicit formula representing the number of matchsticks in any pattern.



7. In a picture on a computer, the height of a building is 50 mm.
   By clicking on a button, the picture size can be enlarged by a factor of 1,2 so that the height of the building in the picture becomes 60 mm.
   (a) Write down a recursive formula that models the above situation.
   (b) Hence, determine the height of the building on the computer after the sixth click of the button.

2. The accompanying graph represents the populations of mice and owls in a field. Use the graph to answer the questions that follow.



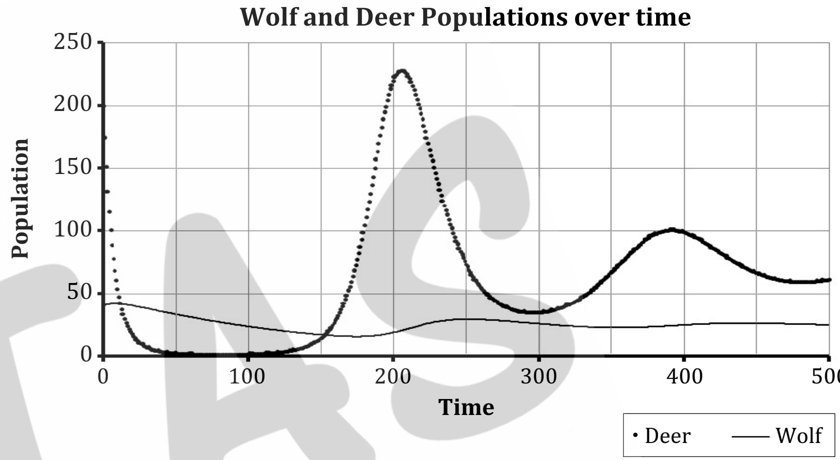
2.1 Give the approximate range of the mouse (prey) population, as shown in this time period.

2.2 State whether the population graphs imply that the populations reach an equilibrium, or if the populations remain cyclic. Give a brief reason for your answer.

2.3 How many months later does the owl population peak after the mice population?

2.4 Is the initial mice population substantially above the carrying capacity of the area? Briefly explain your answer.

3. The graph below represents the relationship between wolf and deer populations for the first 500 cycles. Ignore the possibility of the deer dying out due to a low population for $50 < t < 120$.



3.1 What indicator on the graphs shows that the populations will tend to an equilibrium point?

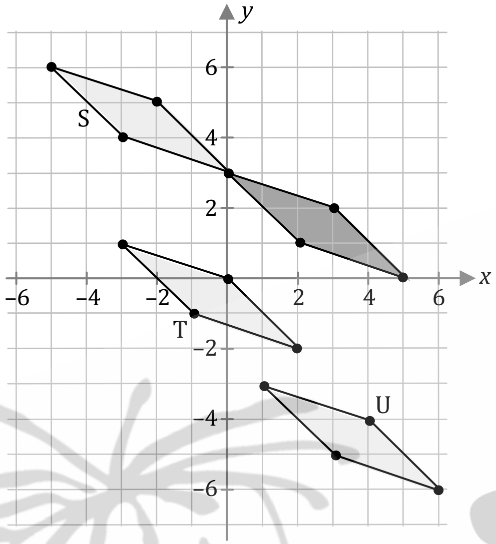
3.2 Estimate the months when the deer population is growing most rapidly.

3.3 State how many months the recovery of the wolf population lags behind the recovery of the deer population.

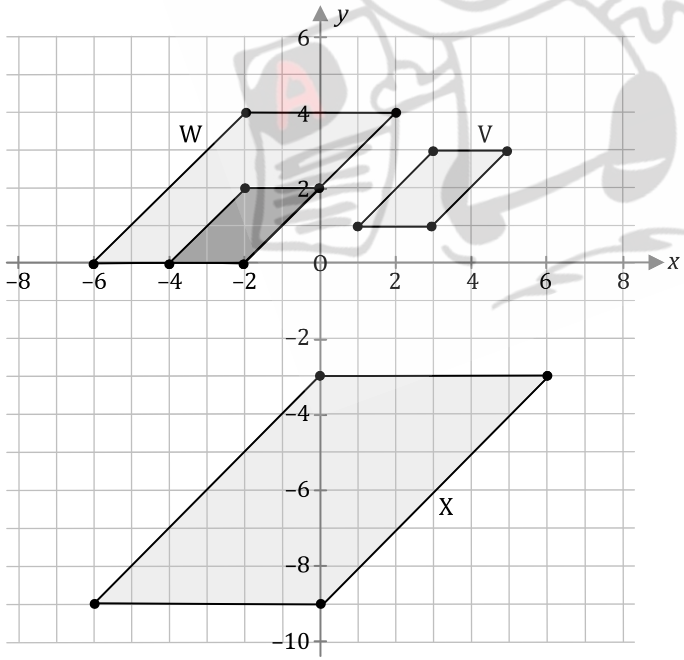
3.4 What causes the deer population to decline so rapidly from its initial population?

3.5 Give the range of the wolf (predator) population.

2. Describe in words, and state the matrix used, to translate the dark figure into the images S, T and U.



3. For each of the images V, W and X, the dark figure has first been translated and then enlarged. Describe in words, and state the matrices used, to effect these transformations.



Exercise MAT 2.2 (Calculating Transformations)

(Solutions on Mat.50)

1. Transform triangle PQR with vertices P(−1; −1), Q(2; 0) and R(0; 3) by:
   (a) an enlargement, by a factor of 3
   (b) a stretch, with a factor of −2, invariant to the *x*-axis
   (c) a shear of factor 4, invariant line the *y*-axis

2. Reflect the given point, as required:
   (a) $(\sqrt{3}; 1)$ in the line $y = \sqrt{3}x$ (b) (3; −1) in the line $y = \frac{1}{2}x$

3. Rotate the given point about the origin, as required:
   (a) (10; 15) rotated by 53,13° (b) $(\sqrt{3}; -2)$ rotated by 30°

4. A triangle with vertices R(13; 0), S(−13; −26) and T(−52; 39) is transformed in three different ways.
   Determine the coordinates of the image after:
   (a) a shear of factor 2, with the *y*-axis as invariant line.
   (b) a rotation about the origin by an angle of 22,62°.
   (c) a reflection about the line $y = \frac{1}{2}x$

5. KYTO is a kite with vertices K(−1; 3), Y(−4; 4), T(−3; 1) and O(0; 0). KYTO is to be enlarged by a factor of 3 through the origin; and then stretched with factor of $\frac{1}{3}$, using the *x*-axis as the invariant line.
   (a) Write down a single matrix that can be used to effect both transformations in the required order simultaneously.
   (b) Hence determine the coordinates of K'Y'T'O', the image of KYTO, using the required transformation.

6. Determine a single matrix that would reflect a figure about the line $y = x$, followed by a shear of factor 2, with the *x*-axis as invariant.

7. Find the single transformation matrix which first enlarges with a factor of −2, and then reflects the figure about the line $y = \sqrt{3}x$.

INVERSE MATRICES BY $[A:I \rightarrow I:A^{-1}]$

Having mastered the art of Gaussian Reduction, let's put it to good use by finding the inverses of square matrices, by using Row Reductions. The process makes use of the identity $[\mathbf{A:I} \rightarrow \mathbf{I:A^{-1}}]$**.** The end result will be that the original matrix is written in **Reduced Row Echelon Form**.

Finding the inverses of smaller matrices (e.g. 2 x 2) is simple, but let's pretend we have no knowledge of finding inverses. This will help establish a pattern common to finding inverse matrices for all matrices, irrespective of their dimension.

The next three examples are of increasingly larger dimension matrices; once you've got the basic principle, you can start tweaking the process to simplify matters for yourself.

Worked Example 4

Inverse of a 2 x 2 matrix: $\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$

Step 1: Set up the matrix, together with the identity matrix.

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array}$$

Step 2: Keep one row unchanged; do row reduction on the other row.

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 17 & -3 & 2 \end{array}\right) \quad \begin{array}{l} \text{Row 1 unchanged} \\ 2 \times \text{Row 2} \; - \; 3 \times \text{Row 1} \end{array}$$

Step 3: Keep the second row unchanged; do row reductions to obtain a second column of zeroes, apart from on the leading diagonal.

$$\left(\begin{array}{cc|cc} 34 & 0 & 8 & 6 \\ 0 & 17 & -3 & 2 \end{array}\right) \quad \begin{array}{l} 17 \times \text{Row 1} \; + \; 3 \times \text{Row 2} \\ \text{Row 2 unchanged} \end{array}$$

Step 4: Express the leading diagonal with equal elements.

$$\left(\begin{array}{cc|cc} 17 & 0 & 4 & 3 \\ 0 & 17 & -3 & 2 \end{array}\right) \quad \begin{array}{l} \text{Row 1} \div 2 \\ \text{Row 2} \end{array}$$

Step 5: Reduce the original matrix to the identity matrix.

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{17} & \frac{3}{17} \\ 0 & 1 & -\frac{3}{17} & \frac{2}{17} \end{array}\right) \quad \begin{array}{l} \text{Row 1} \div 17 \\ \text{Row 2} \div 17 \end{array}$$

The inverse matrix of

$$\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} \text{ is } \begin{pmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{pmatrix} = \mathbf{\frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}}$$

With the shortcut from Grade 10, we obtain the same result – just quicker! Test that the new matrix is the inverse matrix, by multiplying it with the original. Don't forget: the scalar unit fraction is part of the inverse.

Worked Example 5

Inverse of a 3 x 3 matrix: $\begin{pmatrix} 3 & 6 & 8 \\ -1 & 5 & 2 \\ 4 & -8 & 1 \end{pmatrix}$

Step 1: Set up the matrix, together with the identity matrix.

$$\left(\begin{array}{ccc|ccc} 3 & 6 & 8 & 1 & 0 & 0 \\ -1 & 5 & 2 & 0 & 1 & 0 \\ 4 & -8 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}$$

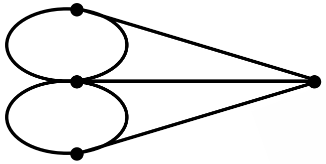
Step 2: Keep one row unchanged; do row reductions on the other rows.

$$\left(\begin{array}{ccc|ccc} 3 & 6 & 8 & 1 & 0 & 0 \\ 0 & 21 & 14 & 1 & 3 & 0 \\ 0 & -48 & -29 & -4 & 0 & 3 \end{array}\right) \quad \begin{array}{l} \text{Row 1 unchanged} \\ 3 \times \text{Row 2} \; + \; \text{Row 1} \\ 3 \times \text{Row 3} \; - \; 4 \times \text{Row 1} \end{array}$$

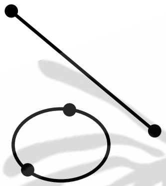
Chapter 4: Gr 10 GRAPH THEORY

TERMINOLOGY AND CONCEPTS

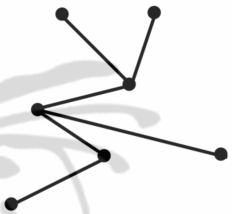
A **graph** is a set of **vertices** and **edges;** every edge starts and ends at a vertex.



Graph A



Graph B



Graph C

A **connected graph** has all vertices directly or indirectly connected to each other. Graphs A and C are connected.

Simple graphs have no loops or multiple edges. Only Graph C is simple.

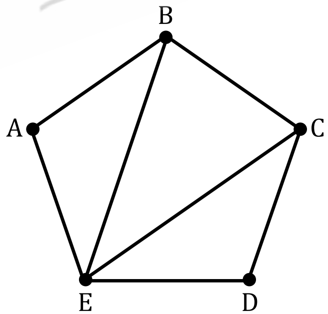
A **tree** is a connected graph, with **no circuits**. Graph C is a tree.

An **undirected** graph means that the weight A → B = B → A
If a loop or a multiple edge is present, the graph is then directed.

The **weight** of an edge is the distance, time, cost or any other factor ascribed to it.

The **order** of a graph is the number of vertices in the graph. Graph D has order 5.

The **degree** of a vertex is the number of edges leading to/from that vertex.



Graph D

A **regular graph** has all vertices of the same degree. Graph D is **not** regular, as the vertices have different degrees.

The **size** of a graph is the total number of edges in the graph. Graph D has size 7.

Adjacent vertices are directly connected to each other, that is, joined by a common edge.

The **neighbourhood** of a vertex are those vertices to which it is directly connected. In Graph D, the neighbourhood of vertex C is B, D and E.

A **path** is a route between two specified vertices, so that no edge is used more than once; not all edges of the graph need be used and the starting point is not usually the endpoint. In Graph D, a possible path from A to D is: A, B, E, C, D. Another path is: A, B, C, D.

A path that starts and ends at the same vertex is a **closed path,** or a **circuit**. Starting at a different vertex and following the same path does **not** constitute a new circuit; rather this is seen as just a different **itinerary** of the same circuit, e.g. A – B – C – D – A is the same circuit as B – C – D – A – B and C – D – A – B – C and D – A – B – C – D

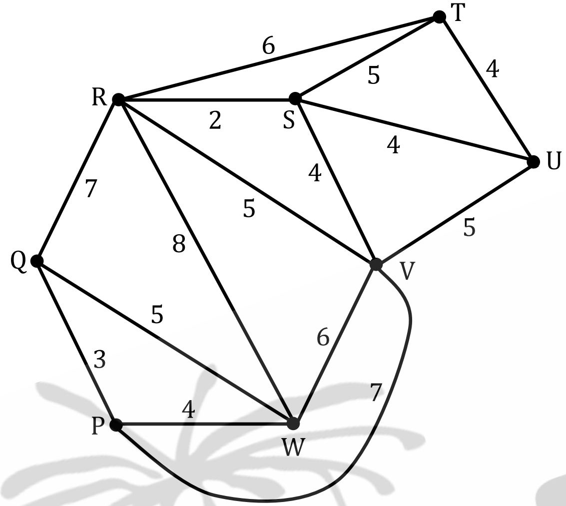
The **girth** of a circuit is the sum of the weight of its edges.

Planar graphs can be drawn without edges crossing each other.

In any graph, the **number of edges** is half the sum of the degrees of the vertices.

Graphs can also be represented in **adjacency matrices,** or as **geometric figures** of nets.

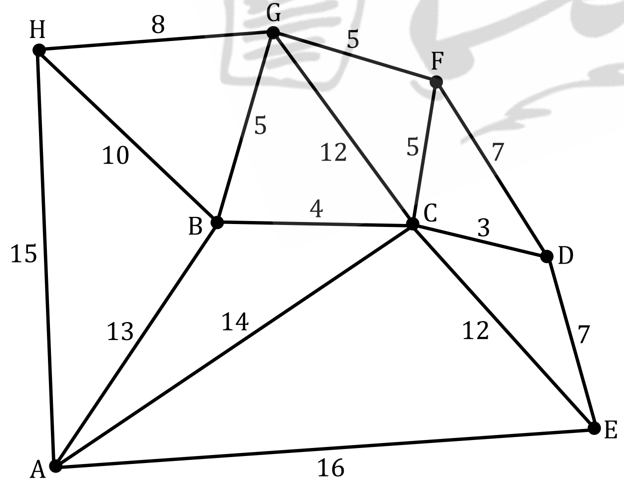
2. The graph alongside represents underground electric cables (the edges) between various attractions (the vertices) at a theme park. The weight of the edges represents the cost (in thousands of rands) of laying the cables between the attractions.



(a) Determine a spanning tree of minimum cost for the cables, based on Kruskal's Algorithm. Clearly indicate the order in which edges are chosen, and the minimum cost of the tree.

(b) Determine a spanning tree of minimum cost for the cables, based on Prim's Algorithm, and starting at vertex W. Clearly indicate the order in which edges are chosen, and the minimum cost of the tree.

3. In the graph below the vertices represent security cameras stationed at various sites.

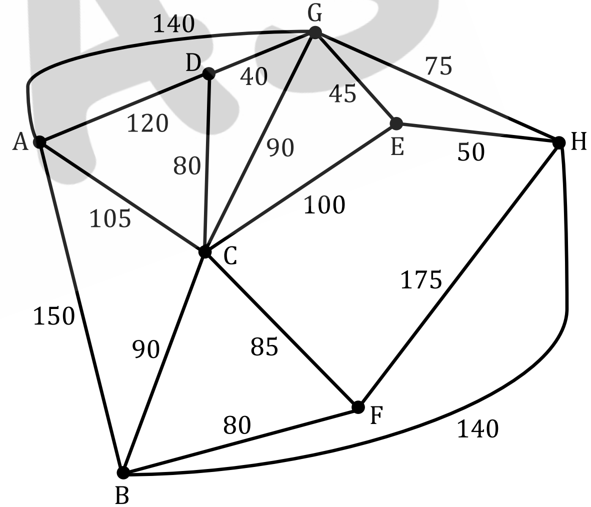


(a) A student has sought to determine a minimum spanning tree, using Kruskal's Algorithm. The edges he has chosen are recorded in order below:

CD (3), BC (4), FC (5), GF (5), BG (5), DE (7), HG (8)
Identify and briefly explain two mistakes he has made.

(b) Hence correctly design a spanning tree of minimum weight, based on Prim's Algorithm, and starting at vertex C. Clearly indicate the order in which edges are chosen, and the minimum weight of the tree.

4. Olive is a sales rep living in City B. She uses a mileage chart to plan a trip, taking in every city and then returns home.



(a) Determine a Lower Bound for her route, initially excluding City A. Use a method based on Kruskal's Algorithm. Carefully record the order of the edges chosen.

(b) Determine an Upper Bound for her route, using the Nearest Neighbour Algorithm. Carefully record the order of the edges chosen.

(c) By inspection find a "good" route for her to take.